Continuous Computational Social Choice

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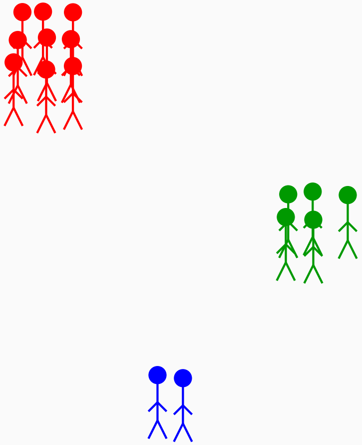

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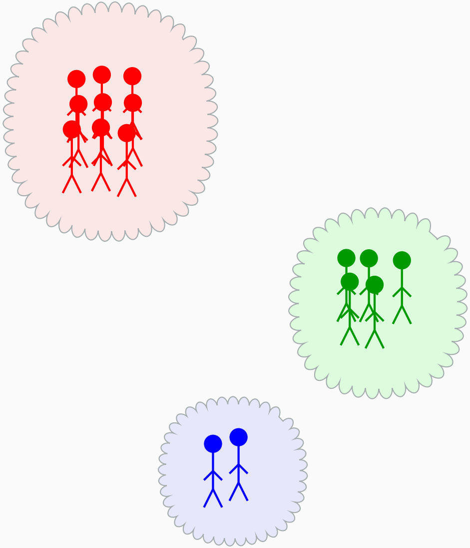

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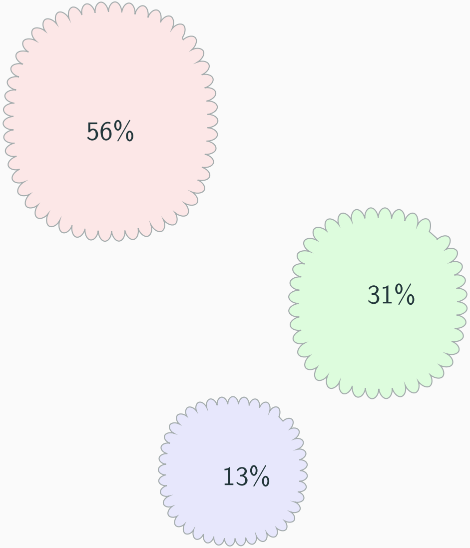

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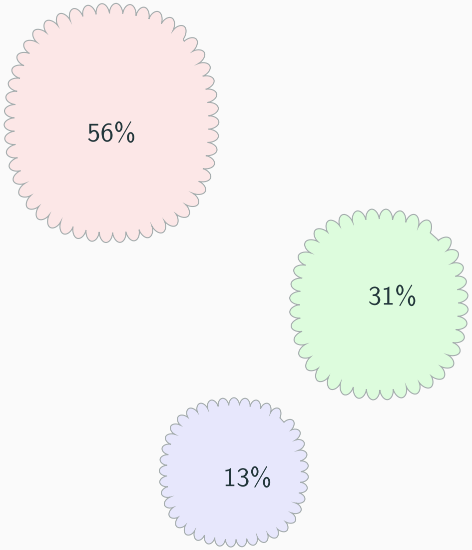
- **Analogous to:** Mean-field Theory
  - Statistical Physics
  - Mean-field Game Theory
  - “Geometry of Voting”



Case Study: Voting & Bribery

Candidates: $\blacktriangle$, $\blacksquare$, and $\bigstar$.

People: preference (e.g. $\blacksquare \succ \blacktriangle \succ \bigstar$)

Society: how many people of which type $\Rightarrow$ **Society Graph:**

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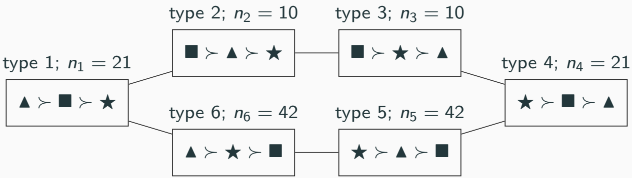

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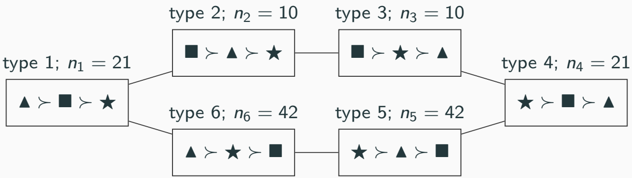

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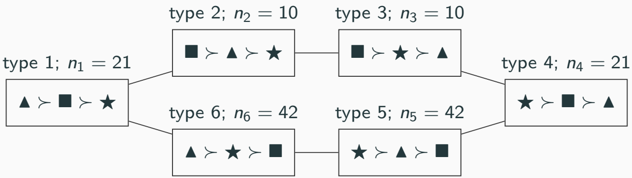

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Continuous Society:

$$\mu = \frac{\mathbf{n}}{\|\mathbf{n}\|_1} \approx (.14, .07, .07, .14, .29, .29)$$

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$$0 \leq x_i \leq n_i \qquad i \in [\tau]$$

$$\sum_{i: c^\star >_i c'} (n_i - x_i) > \sum_{i: c' >_i c^\star} (n_i - x_i) \quad \forall c' \neq c^\star$$

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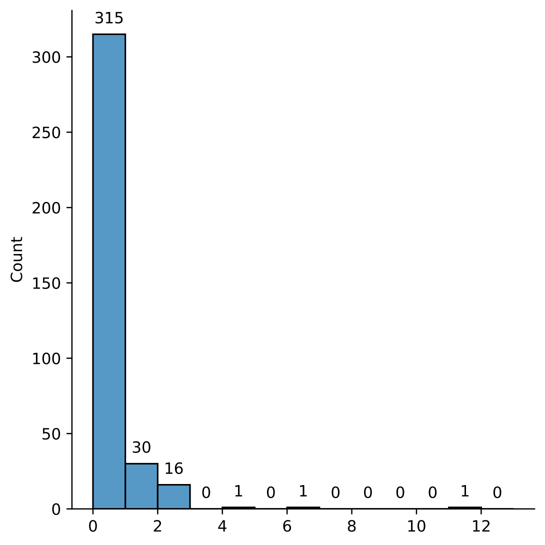
Young Voting: Preflib (Political Elections)

On political elections of PrefLib ($n = 364$):

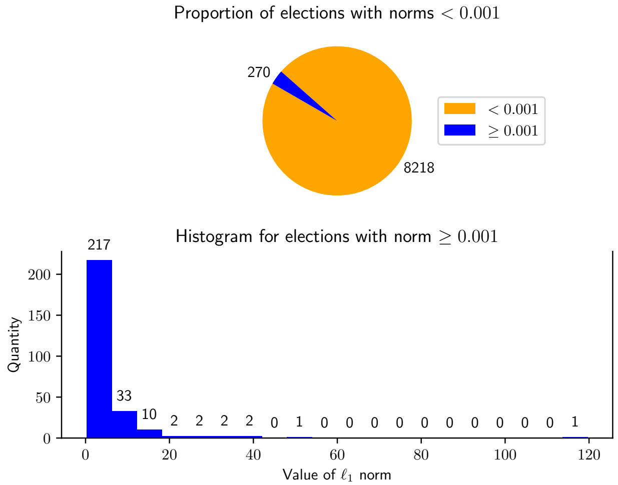
- Young Score vs Young Score$_\infty$ **always** give the same ranking
- On $n = 315$ elections both scores **agree completely**
- On remaining 49 elections never differ by more than 12, or 0.14% in relative terms.



Young Voting: Preflib (All Elections)

On all elections of PrefLib ($n = 8482$ elections):

- Young Score vs Young Score$_\infty$ give the same ranking on 97% instances
- On remaining elections does not differ much



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- We're lucky: shifts up suffice, o/w $\Theta(m!)$ "output types" to consider

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Proof Sketch:

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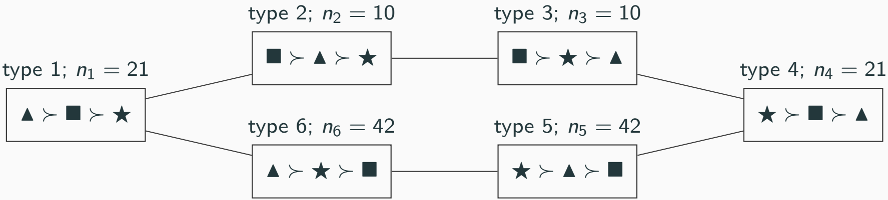
- : Down-scaling weights by a scalar doesn't change the average!

Bribing



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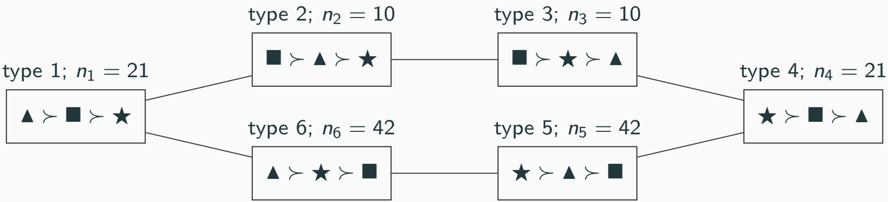

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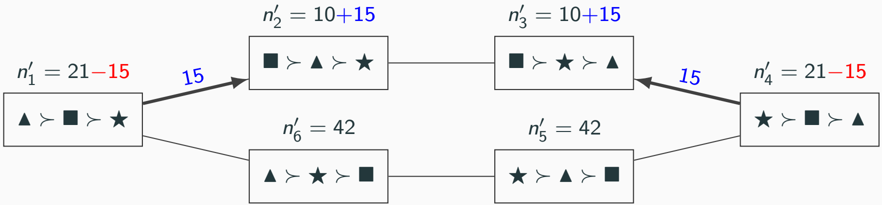

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move $\mathbf{m} = (0, \ldots, 0, +15, +15, 0, \ldots, 0)$ (arc space of complete oriented graph)

change $\boldsymbol{\Delta} = \Delta(\mathbf{m}) = (-15, +15, +15, -15, 0, 0)$

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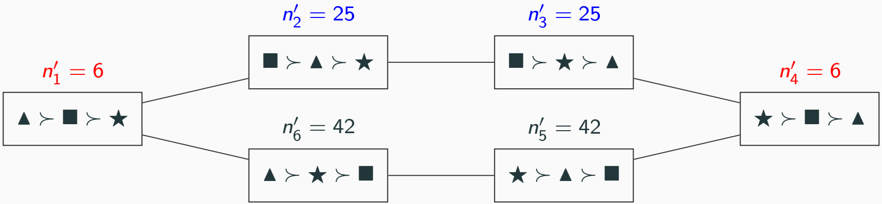

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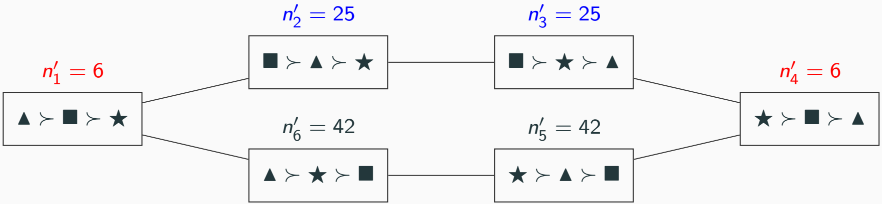

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Actually: Bribery, \$Bribery, Shift Bribery, Swap Bribery, CCDV, etc.

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  - For our costs, optimal face of a known LO relaxation is integral!

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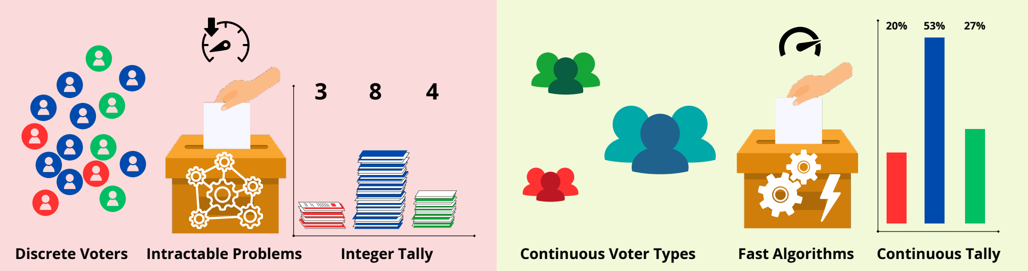

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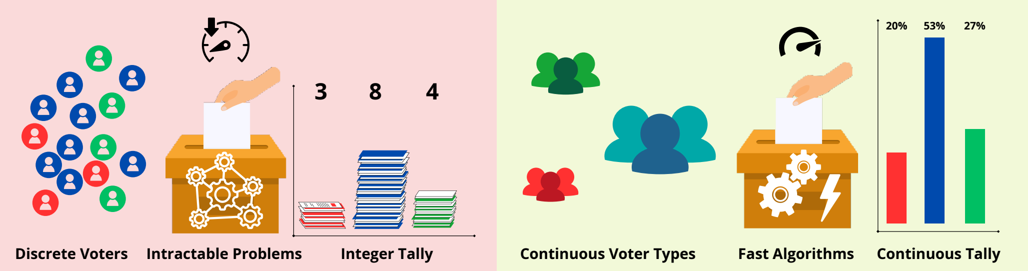

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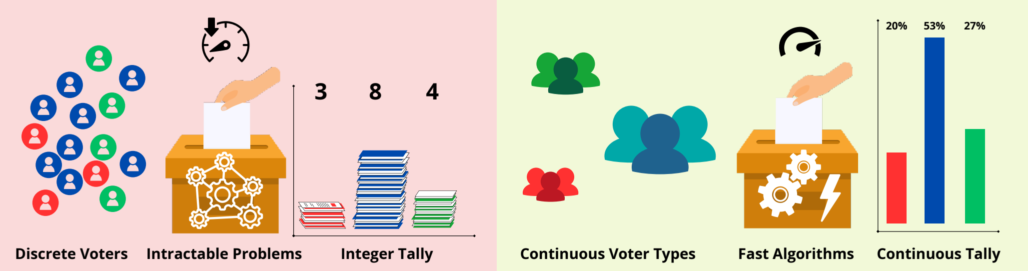

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- Opportunity to use new tools (in ComSoC)

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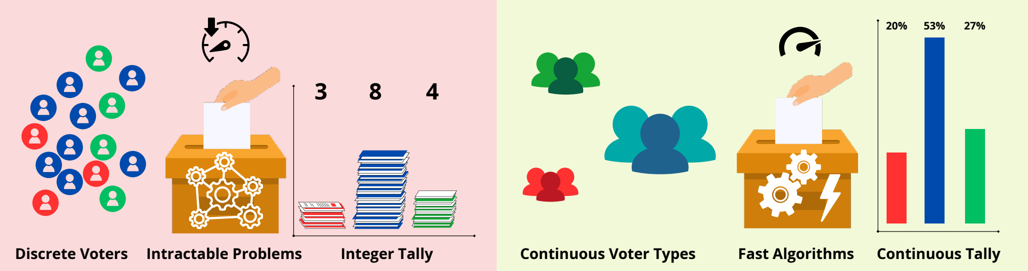

- Est. **half** relevant papers study a problem w/ natural continuous analogue
- Rich and non-trivial new complexity landscape
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- May reveal where hardness is a **modeling artifact**

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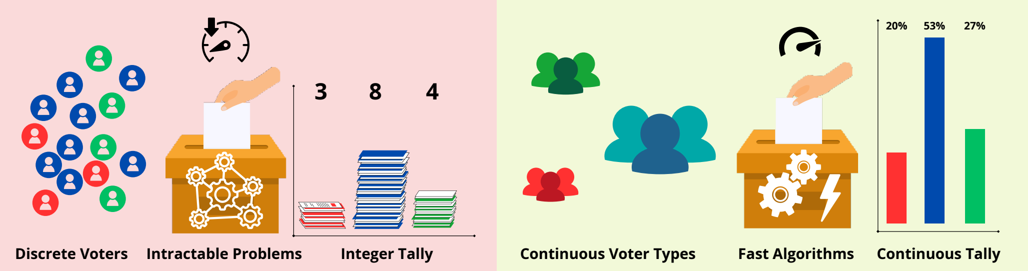

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Thank You!